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RATIONAL SURFACES HAVING ONLY A FINITE NUMBER OF EXCEPTIONAL CURVES

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Abstract

We characterize the rational surfaces X which have a finite number of (-1)-curves under the assumption that $-K_X$ is nef (i.e., the intersection number of K_X with any effective divisor on X is less than or equal to zero, where K_X is a canonical divisor on X) and having self-intersection zero. A (-1)-curve is a smooth rational curve of self-intersection -1.

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1 Introduction

By a surface we mean here a compact complex analytic manifold of complex dimension two. According to the curves on surfaces, one can distinguish between three classes: there are those which have only a finite number of curves, i.e. the surfaces having algebraic dimension zero; those which have curves but not enough, i.e. the surfaces having algebraic dimension one; and those which are rich in curves, i.e. the projective ones. For more details see [5, Theorems 3.1, 4.1, 4.2, 4.3, 5.1], [6], and [12].

We restrict ourselves to curves which are smooth, rational and of self-intersection -1 (such curves are called (-1)-curves), and ask which surfaces have an infinite number of (-1)-curves. The answer is that only rational surfaces may have an infinite number of (-1)-curves (e.g., see [11]).

In this paper we give a characterization of rational surfaces which have a finite number of (-1)-curves under the assumption that an anti-canonical divisor of the surface is nef (see the definition in the next paragraph) and of self-intersection zero.

Let X be a smooth projective rational surface. From now on we assume that $-K_X$ is nef (i.e., the intersection number of the divisor K_X with any effective divisor on X is less or equal to zero, where K_X is a canonical divisor on X) and of self-intersection zero.

It is easy to see that X is a blowing-up nine points (possibily infinitely near) of the complex projective plane.

According to the position of the nine points, X may have a finite or an infinite number of (-1)-curves.

Masayoshi Nagata ([10], proposition 6a., p.282) proved that if the nine points are in general position, then X has an infinite number of (-1)-curves (i.e., smooth rational curves of self-intersection -1).

Ulf Persson and Rick Miranda ([9]) studied the case when the nine points are the base points of a linear system of plane cubics without fixed components. In this case X is an elliptic surface with a section. They classified all such surfaces which have a finite number of (-1)-curves and called them *extremal jacobian elliptic rational surfaces*. For each case, they gave the number of (-1)-curves.

We will use the following notations:
$\sim$ the linear equivalence of divisors on X;
$[D]$ the set of divisors D' on X such that $D' \sim D$;
$Div(X)$ the group of divisors on X;
$NS(X)$ the group quotient $\dfrac{Div(X)}{\sim}$ of $Div(X)$ by $\sim$ (the linear, algebraic and numerical equivalences are the same on $Div(X)$ since X is a rational surface);
$D.D'$ will denote the intersection number of the divisor D with the divisor D', in particular the self-intersection of D is $D^2 = D.D$;

$\overline{D}$ the associated element to the divisor D in the tensor product of the group $NS(X)$ with the field of rational numbers over the ring of integers.

Following [9], we will define a smooth projective rational surface having a finite number of (-1)-curves on it *an extremal rational surface.* Our main result gives a classification of extremal rational surfaces:

Theorem 1.1 *Let X be a smooth projective rational surface having $-K_X$ nef and of self-intersection zero. Then the following are equivalent:*

1. *X is extremal.*

2. *X satisfy the two conditions below:*

   a. *the rank of the matrix $(C_i.C_j)_{i,j\ =1,\ldots,r}$ is equal to 8, where $\{C_i\ ; i = 1, \ldots, r\}$ is the finite set of (-2)-curves on X; a (-2)-curve is a smooth rational curve of self-intersection -2.*

   b. *There exist r strictly positive rational numbers a_i $, i = 1, \ldots,$ r such that $-\overline{K}_X = \sum_{i=1}^{i=r} a_i \overline{C}_i$.*

This paper is organized as follows. In most cases X denotes a smooth projective rational surface defined over the field of complex numbers $\mathbb{C}$ such that its anti-canonical class is nef and of self-intersection zero. In Sect. 2 we introduce some well-known facts about smooth rational surfaces. In Sect. 3 we prove that the existence of a family of (-2)-curves on X such that its elements are linearly dependents in the tensor product $NS(X)_{\mathbb{Q}}$ of the Néron-Severi group $NS(X)$ of X and the field of rational numbers $\mathbb{Q}$ over the ring of integers $\mathbb{Z}$ gives an elliptic structure to X (see proposition 3.5, p.10); As a corollary, the number of (-2)-curves on X is finite and bounded by the optimal integer 12 (see corollary 3.6, p.10). In Sect. 4 we give a useful criterion (see proposition 4.2, p.12) for the existence of an infinite number of (-1)-curves on the surface. In Sect. 5 we give the proof of theorem 1.1.

2 Preliminaries

In this section we fix our notations and gather some well-known general facts concerning smooth projective rational surfaces.

2.1 Notation and First Properties

Let X be a smooth projective rational surface defined over the field of complex numbers $\mathbb{C}$. Recall that the Néron-Severi group $NS(X)$ of X is the group of divisor classes on X modulo algebraic equivalence. The following facts are well-known:

- $NS(X)$ is finitely generated abelien group of rank $\rho(X)$ ($\rho(X)$ is called the Picard number of X);
- $NS(X)$ is torsion free abelien group;
- $\rho(X) = 10 - K_X^2$, where K_X represent a canonical divisor on X;
- $NS(X)$ is equipped with a symmetric bilinear form induced from the intersection form defined on the set of divisors $Div(X)$ on X; this symmetric bilinear form will be called from now on the intersection form on $NS(X)$;
- $NS_{\mathbb{Q}}(X)$ is by definition the tensor product of $NS(X)$ and the field of rational numbers $\mathbb{Q}$ over the ring of integers $\mathbb{Z}$,i.e., $NS_{\mathbb{Q}}(X) = NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$;
- $NS_{\mathbb{Q}}(X)$ is equipped with a symmetric bilinear form induced by the intersection form on $NS(X)$; this symmetric bilinear form will be called from now on the intersection form on $NS_{\mathbb{Q}}(X)$.

If D is a divisor on X, we adopt the following notations:

- $[D]$ its equivalence class in $NS(X)$;
- $\overline{D}$ is the unique element associated to $[D]$ in $NS_{\mathbb{Q}}(X)$.

By definition:

- K_X (resp. $-K_X$) denotes a canonical (resp. anti-canonical) divisor on the surface X ;
- $[K_X]$ (resp. $[-K_X]$) is the canonical (resp. anti-canonical) class of X;
- $K_X^{\perp}$ is the orthogonal of $\overline{K}_X$ in $NS_{\mathbb{Q}}(X)$, and by abuse of notation, the orthogonal of $[K_X]$ in $NS(X)$ is also denoted by $K_X^{\perp}$.

The intersection form defined on $Div(X)$ (resp. on $NS(X)$, resp. on $NS_{\mathbb{Q}}(X)$) is noted by "$.$". In particular for any two divisors D and D' on X, the equalities hold: $D.D' = \overline{D}.\overline{D'} = [D].[D']$; if $D = D'$, the intersection number of D and D' is called the self-intersection of the divisor D and is denoted by D^2.
Similarly for every element x of $NS_{\mathbb{Q}}(X)$, the rational number $x.x$ is denoted by x^2 and we say that it is the self-intersection of x.

The following well-known lemma is useful:

Lemma 2.1 *Let X be a smooth projective rational surface such that the self-intersection of its canonical divisor K_X is zero. Then the intersection form is negative semi-definite on $K_X^{\perp}$; and for every element $x \in K_X^{\perp}$, the following equivalence hold:*

$$x^2 = 0 \textit{ if and only if } x = r\overline{K}_X \textit{ for some rational number } r.$$

We give here an elementary proof. Let x be an element of $K_X^{\perp}$, we can suppose that x is the divisor class of a divisor D on X. If D^2 is strictly larger than zero, then by using the index theorem ([4]), K_X would be either of self-intersection strictly less that zero or numeriquely trivial. these are impossible since X is rational and $K_X^2 = 0$.
Let A be an ample divisor on X, and consider the divisor E defined as $E = A.K_X D - A.DK_X$. We have $E^2 = 0 = A.E$.
Hence, by the index theorem ([4]), the divisor $A.K_X D$ is numeriquely equivalent to $A.DK_X$. Since X is rational and $-K_X$ is an effective divisor, we obtain that $\overline{D} = \dfrac{A.D}{A.K_X}\overline{K_X}$.

2.2 Riemann-Roch theorem for Rational Surfaces

Let X be a smooth projective surface, its canonical divisor is denoted by K_X. If D is a divisor on X, then:

- $O_X(D)$ is the invertible sheaf associated to D;
- for $i = 0, 1, 2$: $H^i(X, O_X(D))$ is the i^{th} cohomology group of the invertible sheaf $O_X(D)$;
- for $i = 0, 1, 2$: $h^i(X, O_X(D))$ is the dimension of $H^i(X, O_X(D))$.

Definition 2.2 *A divisor D on a smooth projective surface X is effective if $n_i \geq 0$ for all $i = 1, \ldots, r$, where r is a non-negative integer and where $D = n_1C_1 + \ldots + n_rC_r$ (the C_i, $i = 1, \ldots, r$ are reduced irreducible curves on X).*
An element of $NS(X)$ is effective if it is the class of an effective divisor on X.

A proof of the next lemma is given in [4, proposition 7.7, p. 157]

Lemma 2.3 *An elemnet $[D]$ of $NS(X)$ is effective if and only if the integer $h^0(X, O_X(D))$ is not equal to zero.*

It is easy to see that:

Lemma 2.4 *If D is an effective divisor on a smooth projective rational surface X, then the integer $h^2(X, O_X(D))$ is equal to zero.*

Definition 2.5 *Let D be a divisor on a smooth projective surface X. The Euler-Poincaré characteristic $\chi(D)$ of D is the following integer:*

$$h^0(X, O_X(D)) - h^1(X, O_X(D)) + h^2(X, O_X(D)),$$

where $O_X(D)$ is the invertible sheaf associated to D. This integer is also called the Euler-Poincaré characteristic of the sheaf $O_X(D)$ (resp. of the class $[D]$ of D in the Néron-Severi group $NS(X)$ of X) associated to the divisor D.

The Riemann-Roch theorem is:

Theorem 2.6 *Let D be a divisor on a smooth projective rational surface X, the Euler-Poincaré characteristic $\chi(D)$ of the divisor D is given by:*

$$\chi(D) = 1 + \frac{1}{2}(D^2 - K_X.D)$$

Definition 2.7 *A smooth projective rational surface is anti-canonical if the anti-canonical class of this surface is effective.*

It is easy to see:

Lemma 2.8 *Every smooth rational surface for which the self-intersection of its canonical class is larger or equal to zero is anti-canonical.*

The adjunction formula is:

Lemma 2.9 *Let Y be a smooth projective surface. For every irreducible curve C on Y, the equality hold:*

$$C^2 + C.K_Y = 2p_a(C) - 2,$$

where K_Y is a canonical divisor on Y and $p_a(C)$ is the arithmetic genus of the curve C.

Definition 2.10 *Let X be a smooth projective surface.*

- *A divisor D on X is nef if and only if $D.C \geq 0$ for every irreducible curve C on X.*
- *An element of $NS(X)$ is nef if it is the class of a nef divisor on X.*

Using the Riemann-Roch theorem and the adjunction formula, one can obtain:

Lemma 2.11 *Let X be a smooth projective rational surface such that its anti-canonical class is nef.*
The self-intersection of a reduced irreducible curve C on X of self-intersection non-positive is either -2 or -1; and the following hold:

- *$C^2 = -2$ if and only if C is a smooth rational curve of self-intersection -2.*
- *$C^2 = -1$ if and only if C is a smooth rational curve of self-intersection -1.*

If T is a reduced irreducible curve on the smooth rational surface X such that T is not orthogonal to canonical divisor on X, then the following properties are valid:

Lemma 2.12 *With the same notations as above.*

i− $NS_{\mathbb{Q}}(X) = \mathbb{Q}\overline{T} \oplus K_X^{\perp}$, *in particular if* T *is a smooth rational curve of self-intersection* -1 *on* X, *then we have:* $NS(X) = \mathbb{Z}[T] \oplus K_X^{\perp}$;

ii− *the dimension of the* $\mathbb{Q}$*-vector space* $K_X^{\perp}$ *is* $9 - K_X^2$.

Proof. [$i-$] Let $x \in NS_{\mathbb{Q}}(X)$:

− if $x \in \mathbb{Q}\overline{T} \cap K_X^{\perp}$, we have:

$$\begin{cases} \text{x}.\overline{K_X}{=}0; \\ x{=}\alpha\overline{T}, \qquad \text{for some } \alpha \in \mathbb{Q}. \end{cases}$$

Hence $0 = \alpha\overline{T}.\overline{K_X}$ and consequently $\alpha = 0$.
It follows that $\mathbb{Q}\overline{T} \cap K_X^{\perp} = \{0\}$;

− on the other hand, we have:

$$x - \frac{\overline{K_X}.x}{K_X.T}\overline{T} \in K_X^{\perp},$$

since
$\overline{K_X}.(x - \dfrac{\overline{K_X}.x}{K_X.T}\overline{T}) = \overline{K_X}.x - \dfrac{\overline{K_X}.x}{K_X.T} K_X.T = \overline{K_X}.x - \overline{K_X}.x = 0.$
In conclusion, $NS_{\mathbb{Q}}(X) = \mathbb{Q}[T] \oplus K_X^{\perp}$.

[$ii-$] is a consequence of [$i-$] and the fact that $\rho(X) = 10 - K_X^2$.

2.3 Some linear Algebra Results

We give here some linear algebra results that we will use in section.

Definition 2.13 *Let* n *be a non-negative integer. Consider two* n*-tuple* $Z(t) = (z_1(t), \ldots, z_n(t))$ *and* $Z'(t) = (z'_1(t), \ldots, z'_n(t))$ *of real numbers (resp.,* $Z(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix}$ *and* $Z'(t) = \begin{pmatrix} z'_1(t) \\ \vdots \\ z'_n(t) \end{pmatrix}$ *of real numbers.*

We write $Z(t) \leq Z'(t)$ *if and only if the inequality* $z_i(t) \leq z'_i(t)$ *hold for all* $i = 1, \ldots n$.

Definition 2.14 *Let* n *be a non-negative integer, A family* $(Z(t))_{t\in I}$ *of* n*-tuple* $Z(t) = (z_1(t), \ldots, z_n(t))$ *(where* I *is non empty set) with coefficients in* $\mathbb{Q}$*, or in* $\mathbb{R}$*, is bounded if there exists some scalar* b *such that for every* i *in* $\{1, \ldots, n\}$ *and for every* t *in* I*, the inequality hold:* $|z_i(t)| \leq b$*, where* $|x|$ *denotes the absolute value of the scalar* x.

The next lemma is elementary and the proof is omitted.

Lemma 2.15 *Let* $A = (a_{i,j})$ *be an invertible matrix of order* n *with coefficients in* $\mathbb{Q}$*, or in* $\mathbb{R}$*. For every family* $(Z(t))_{t\in I}$ *(where* I *is a non empty set), the following assertions are equivalents:*

1. *the family* $(Z(t))_{t\in I}$ *is bounded.*

2. the family $(AZ(t))_{t\in I}$ is bounded.

Corollary 2.16 *Let $Z(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix}$ be a family of real numbers. and let $A = (a_{i,j})$ be a square matrix of order n with coefficients in $\mathbb{Q}$, or in $\mathbb{R}$, having the following shape:*

$$A = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right) \qquad \text{(where } D \text{ is an invertible matrix of order } n-1\text{)}.$$

1. *If the family $(Z(t))_{t\in I}$ is bounded, then $(AZ(t))_{t\in I}$ is bounded too.*

2. *The converse is true if there exist some scalares $d_1, \dots, d_n$ such that:*

   a. *$d_n \neq 0$;*

   b. *for every t in I, the following equality hold:*

   $$z_1(t)^2+\cdots+z_{n-1}(t)^2-\sum_{1\leq i<j\leq n-1} a_{i,j}z_i(t)z_j(t) = d_1z_1(t)+\cdots+d_{n-1}z_{n-1}(t)+d_nz_n(t).$$

Proof. Let $Y(t) = AZ(t)$. Then

$$Y(t) = \left(\begin{array}{c} DZ'(t) \\ \hline 0 \end{array}\right)$$

If we write for every $t \in I$, $Y(t)$ as $Y(t) = \begin{pmatrix} Y'(t) \\ y_n(t) \end{pmatrix}$ and $Z(t)$ as $Z(t) = \begin{pmatrix} Z'(t) \\ z_n(t) \end{pmatrix}$, we have for every $t \in I$, $Y'(t) = DZ'(t)$ and $y_n(t) = 0$.
The fact that $(Z(t))_{t\in I}$ is bounded shows that $(Z'(t))_{t\in I}$ is bounded too; Since D is invertible, the lemma 2.15 shows that $(DZ'(t))_{t\in I}$ is bounded, that is $(Y'(t))_{t\in I}$ is bounded, it follows then $(Y(t))_{t\in I}$ is bounded (since $y_n(t) = 0$ for all $t \in I$.

Conversely, if $(Y(t))_{t\in I}$ is bounded, then $(DZ'(t))_{t\in I}$ is bounded too, and by using the lemma 2.15, $(Z'(t))_{t\in I}$ is bounded.

The hypothesis $a.$ and $b.$ confirm that the number $z_n(t)$ is a function of $z_1(t), \dots, z_{n-1}(t)$. Hence the family $(z_n(t))_{t\in I}$ is bounded. In conclusion, $(Z(t))_{t\in I}$ is bounded.

3 Criterion for a Rational Surface to Be Elliptic

Let X be a smooth projective rational surface such that the self-intersection of its canonical divisor is zero. Recall that a (-2)-curve on X is a smooth rational curve of self-intersection -2. In this section we prove that if $\{C_1, \dots, C_k\}$ is a set of (-2)-curves on X such that the vectors $\{\overline{C_1}, \dots, \overline{C_k}\}$ are linearly dependents in $NS_{\mathbb{Q}}(X)$ (where k is a non-negative integer), then the surface X is elliptic. As a corollary, we prove that the number of (-2)-curves on X is bounded by 12; and this bound is optimal.

Recall some definitions:
let $f : X \longrightarrow B$ be a relatively minimal elliptic surface X over a smooth curve B of genus g. By this we always mean the following:
X is a smooth projective surface defined over $\mathbb{C}$ equipped with an elliptic relatively minimal fibration $f : X \longrightarrow B$. That is, f is a surjective morphism satisfying the following conditions:

1- Almost all the fibres are elliptic curves;

2- Each fibre does not contain a (-1)-curve as a component.

If $D = n_1C_1 + \cdots + n_pC_p$ is the decomposition of the divisor D in reduced irreducible components $C_1, \ldots, C_p$, then it will be called multiple of multiplicity m if and only if the greatest commun divisor of the integers $n_1, \ldots, n_p$ is equal to m and $m \geq 2$.
It follows that if D is multiple of multiplicity m, then $D = mD'$ where m is the multiplicity of D and D' is a divisor which is not multiple.
We denote by:

- χ the Euler characteristic of the surface X $(\chi = \chi(O_X))$;

- F a general fiber of f;

- F_b the fiber of f over the point $b \in B$;

- $F_{b_1} = m_1F_1, \ldots, F_{b_r} = m_rF_r$ the multiple fibers of f of multiplicity $m_1, \ldots, m_r$ respectively.

Recall that the Néron-Severi group $NS(X)$ of X is the group of divisor classes on X modulo algebraic equivalence. If C is a divisor on X, we denote by $[C]$ its class in $NS(X)$ and by $\overline{C}$ the element associated to $[C]$ in $NS_{\mathbb{Q}}(X)$, where $NS_{\mathbb{Q}}(X) = NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
We have ([1, corollaire 12.3, p. 162]):

$$\overline{K_X} = (2g - 2 + \chi + r - \Sigma_{i=1}^{i=r} \frac{1}{m_i})\overline{F}. \tag{1}$$

We need the following lemma:

Lemma 3.1 *Let $f : X \to B$ be an elliptic fibration of a smooth projective surface X over a smooth curve B. Then the topological character of a reducible fibre is greater than or equal to the number of irreducible components of this fiber. With equality if and only if this fiber is of type I_n (where n is greater than or equal to 2).*

Arnaud Beauville ([3, the lemma of the page 345]), showed that there exist some inequalities between the genus g of the curve B and the irregularity q of the surface X $(q = h^1(X, O_X)$, O_X

is the structure sheaf of X) mainely:

$$g \leq q \leq g+1,$$

In particular if X is rational, then $q=0$ and the genus g of the curve B is necessarily zero. Thus B is isomorphic to $\mathbb{P}^1$. This fact motivates the following definition:

Definition 3.2 *An elliptic rational surface is a relatively minimal elliptic rational surface over $\mathbb{P}^1$.*

The following result is straightforward:

Lemma 3.3 *Let X be an elliptic rational surface.*
The anti-canonical class of X is nef.

The following result is straightforward:

Lemma 3.4 *Let X be an elliptic rational surface.*
The anti-canonical class of X is nef.

This is our main result in this section:

Proposition 3.5 *Let X be a smooth projective rational surface such that the self-intersection of its canonical divisor is zero.*
If there exists a family of (-2)-curves on X such that its elements are linearly dependents in $NS_{\mathbb{Q}}(X)$, then the surface X is elliptic.

Proof. Let $\{C_1, \ldots, C_k\}$ be a family of (-2)-curves on X such that its elements are linearly dependents in $NS_{\mathbb{Q}}(X)$ (k is a non-negative integer). We can suppose that there exist some non-negative integers $a_1, a_2, \ldots, a_k$ such that:

$$\Sigma_{i=1}^{i=r} a_i [C_i] = \Sigma_{j=r+1}^{j=k} a_j [C_j],$$

where r is an integer satisfying $1 \leq r \leq k-1$.
Define C and C' as :
$C = \Sigma_{i=1}^{i=r} a_i C_i$ and $C' = \Sigma_{j=r+1}^{j=k} a_j C_j$.
We have: $0 \geq C^2 = C.C' \geq 0$, then $C^2 = 0 = C'^2$ and C is linearly equivalent to C' .
The complete linear system $|C|$ is without fixed components, without base points and by using Stein factorisation gives an elliptic fibration $f : X \longrightarrow \mathbb{P}^1$ such that f is relatively minimal.

An immediat consequence is:

Corollary 3.6 *Let X be a smooth projective rational surface such that the self-intersection of its canonical divisor is zero.*
If the number of (-2)-curves on X is greater than or equal to ten, then X is elliptic.

Nagata [10, lemme 6.1, p. 292] proved that if X is the blowing-up nine points of the projective plane, then the set of (-2)-curves is finite. Here we give another proof of his result in the more general context of smooth projective rational surfaces X such $K_X^2 = 0$, where K_X is a canonical divisor of the surface. This result is a consequence of the proposition above.

Corollary 3.7 *Let X be a smooth projective rational surface such that the self-intersection of its canonical divisor is zero. Then the number of (-2)-curves is less than or equal to 12, and this bound is optimal.*

Proof. Consider a family $\{C_1, C_2, \ldots, C_k\}$ of (-2)-curves on X, where k is an integer, Two disjoint possibilities may occur:

If the vectors $\overline{C_1}, \overline{C_2}, \ldots, \overline{C_k}$ are linearly independants in $NS_{\mathbb{Q}}(X)$ which is of dimension 10, then $k \leq 9$.

If not, X admits a relatively minimal elliptic fibration $f : X \longrightarrow \mathbb{P}^1$. Recall the equality (see [2] lemme $VI.4$ page 95):

$$\chi_{top}(X) = \chi_{top}(B)\chi_{top}(F) + \Sigma_{b \in B}(\chi_{top}(F)$$

Here, we have:

- $B = \mathbb{P}^1$;
- $\chi_{top}(X)$ is the topological character of X ;
- $\chi_{top}(F)$ is the topological character of F ;
- F_b is the fiber of f over the point $b \in B$;
- $\chi_{top}(F_b)$ is the topological character of F_b pour $b \in B$.

If χ is the Euler characteristic of the surface X, the Noether formula is:

$$12\chi = K_X^2 + \chi_{top}(X) \qquad (\chi = \chi(O_X)).$$

Since $K_X^2 = 0$ and $\chi_{top}(F) = 0$ (since F is an elliptic curve) we have:

$$12\chi = \Sigma_{b \in B}\chi_{top}(F_b).$$

We denote by $F_1, F_2, \ldots, F_t$ the reducible fibers of f. We have for every $i = 1, 2, ..., t$:

$l_i \leq \chi_{top}(F_i)$, where l_i is the number of irreducible components of F_i.

Which gives

$$\Sigma_{i=1}^{i=t} l_i \leq 12\chi.$$

Hence the number of (-2)-curves is less or equal than $12\chi = 12$ (since X is rational).

The number of (-2)-curves of the extremal jacobian elliptic rational surface X_{3333} ([9, Tableau 5.3, page 549]) is 12, which ends the proof.

4 Some Rational Surfaces Having an Infinite Number of (-1)-Curves

In what follows a (-1)-divisor D on a smmoth projective surface Z is a divisor on Z such that $D^2 = -1 = D.K_Z$, where K_Z is a canonical divisor on Z. We need the following result:

Proposition 4.1 *Let X be a smooth projective rational surface such that $K_X^2 = 0$. Suppose that the intersection form is negative definite on the space spanned over the field of rational numbers by the $\overline{C_i}$, $i = 1\ldots, l$, where the C_i are (-2)-curves on X.*
Then there exists an infinite number of (-1)-divisors such that each (-1)-divisor intersect positively each C_i, $i = 1, \ldots, l$.

Proof. The hypothesis "the intersection form is negative definite on the space spanned by the $\overline{C_i}$, $i = 1, \ldots, l$ " shows that the vectors $\{\overline{C_i},\ i = 1, \ldots, l\}$ are linearly independents in the orthogonal of the canonical divisor of X. In particular, l is less than or equal to 8. For a large integer n (e.g., $n \geq E_0.C_j$, for all $j = 1, \ldots, l$, where E_0 is a fixed (-1)-curve), we consider the unique divisor Δ_n whose support is in the set of (-2)-curves C_i, $i = 1, \ldots, l$ and defined by:

$$\Delta_n.C_j = |d|(n - E_0.C_j)$$

for each $j = 1, \ldots, l$, where d is the determinant of the matrix $(C_i.C_j)_{1\leq i,j\leq l}$ ($|d|$ is the absolute value of d).
The divisor E_n defined for large n by:

$$E_n = E_0 + \Delta_n - (2E_0.\Delta_n + {\Delta_n}^2)(-K_X)$$

is a (-1)-divisor intersecting positively each C_i, $i = 1, \ldots, l$. And for $E_n \neq E_m$ for $n \neq m$.

Proposition 4.2 *Let X be a smooth projective rational surface such that the anti-canonical class is nef and of self-intersection zero.*
If the intersection form is negative definite on the space spanned by $\overline{C}_i$, where the C_i constitute a connected component of the set of all (-2)-curves on X, then the number of (-1)-curves on X is infinite.

Proof. By hypothesis we can find a connected component $\{C_1, \dots, C_l\}$ of the family of all (-2)-curves on X such that the intersection form is negative definite on the space spanned by the $\overline{C_i}, i = 1, \dots, l$. An application of the proposition above yields to the existence of an infinite family E_n of (-1)-divisors such that $E_n.C_i \geq 0$. By construction the element E_n is equal to $E_n = E_0 + \Delta_n - (2E_0.\Delta_n + {\Delta_n}^2)(-K_X)$ where the support of Δ_n is in the set $C_i, i = 1, \dots, l$ and E_0 is a fixed (-1)-curve on X.

For each (-2)-curve C different from $C_i, i = 1, \dots, l$, we have $C.C_i = 0$ for all $i = 1, \dots, l$; then $C.E_n = C.E_0 \geq 0$. Consequently E_n intersect every (-2)-curve on X positively. Hence by [7, Theorem, p.3], E_n is irreductible.

5 Proof of Theorem 1.1

Let s be the rank of the matrix $(C_i.C_j)_{i,j=1,\dots,r}$, where $\{C_i; i = 1, \dots, r\}$ is the finite set of (-2)-curves on X.

Suppose that X is extremal, we claim that $s = 8$.
Proof of the claim:

If $s = 0$, i.e., X has no (-2)-curves, then the nine points are in general position and therfore X has an infinite number of (-1)-curves.

If $s \in \{1, \dots, 7\}$, then we can find a divisor D such that the following conditions hold:

1. $D^2 < 0$,
2. $D.K_X = 0$,
3. $D.C_i = 0$, for each $i = 1, \dots, r$.

Then fix a (-1)-curve E_0, and for each integer n, consider the (-1)-divisor E_n defined as follows:

$$E_n = E_0 + nD - (nE_0.D + \frac{n^2}{2}D^2)(-K_X).$$

By [7, Theoerem, p.3] E_n is a (-1)-curve. Two distinct integers n and m give two distinct (-1)-curves E_n and E_m; so X has an infinite number of (-1)-curves.

Hence $s = 8$ is a necessarily condition in order that X to be extremal.

In what follows s is equal to 8. We will distinguish between two cases:

1. **First Case:** If the vector space over the field of rational numbers spanned by the set of (-2)-curves is not equal to the orthogonal of the canonical divisor, then according to the proposition 4.2 (page 12), X will have an infinite number of (-1)-curves.

2. **Second Case**: If the vector space over the field of rational numbers spanned by the set of (-2)-curves is equal to the orthogonal of the canonical divisor, then we will distinguish between the case where X is elliptic or not:

First subcase: If X is not elliptic, then according to proposition 3.5 (10), the (-2)-curves are linearly independents over the field of rational numbers; so the number of (-2)-curves on X is nine. Two cases comes:

- if the set of (-2)-curves is connected, then $-K_X$ is a linear combination of strictly positive integers of all (-2)-curves.
- If the set of (-2)-curves is not connected, then $-\overline{K}_X$ is not a linear combination of strictly positive rational numbers of all (-2)-curves. Applying proposition 4.2 (p. 12) to any connected component of the set of (-2)-curves for which the intersection form is negative definite gives the fact that X contains an infinite number of (-1)-curves.

Second subcase: If X is elliptic, then $-\overline{K}_X$ is a linear combination of strictly positive rational numbers of all (-2)-curves.

In conclusion if X is extremal, then it satisfies the two conditions below:

a. the rank of the matrix $(C_i.C_j)_{i,j\ =1,\dots,r}$ is equal to 8, where $\{C_i\ ; i=1,\dots,r\}$ is the finite set of (-2)-curves on X;

b. There exist r strictly positive rational numbers $a_i\ , i = 1,\dots,\ r$ such that $-\overline{K}_X = \sum_{i=1}^{i=r} a_i\overline{C}_i$.

Conversely, the fact that the rank of the matrix $(C_i.C_j)_{i,j\ =1,\dots,r}$ is equal to eight enables us to suppose for example that the vectors $\overline{C_1},\dots,\overline{C_8}$ are linearly independents in $NS_{\mathbb{Q}}(X)$ and the intersection form (induced by the intersection form on $K_X^{\perp}$) is negative definite on the vector space over the field of rational numbers spanned by the vectors $\overline{C_1},\dots,\overline{C_7}$ and $\overline{C_8}$.
Since $-K_X$ is not trivial and of self-intersection zero, the vectors $\overline{C_1},\dots,\overline{C_8}$ and $\overline{-K_X}$ are linearly independents in $NS_{\mathbb{Q}}(X)$, in particular

- the elements $[C_1],\dots,[C_8]$ and $[-K_X]$ are linearly independents in $NS(X)$;
- the vecor space $<\overline{C_1},\dots,\overline{C_8},\overline{-K_X}>$ is equal to $K_X^{\perp}$.

It is easy to prove the following lemma:

Lemma 5.1 *Let E be a (-1)-curve on X. Then the equality*
$-\overline{K}_X = \sum_{i=1}^{i=r} a_i\overline{C}_i$ *gives rise the inequalities:*

$$0 \leq E.C_i \leq \frac{1}{a_i},$$

for every $i = 1,\dots,r$.

Let E_0 be a fixed (-1)-curve on X and consider a (-1)-curve E on X. There exist some rational numbers $\alpha_1(E),\dots,\alpha_9(E)$ such that:

$$\overline{E} = \overline{E_0} + \alpha_1(E)\overline{C_1} + \cdots + \alpha_8(E)\overline{C_8} + \alpha_9(E)(\overline{-K_X}).$$

We claim that:
the numbers $\alpha_1(E), \ldots, \alpha_9(E)$ may be taken such that their denominators are bounded.
In fact, let M be the lattice of $NS(X)$ defined by the free family
$\{[C_1], \ldots, [C_8], [-K_X]\}$, i.e., $M = \mathbb{Z}\overline{C}_1 \oplus \ldots \oplus \mathbb{Z}\overline{C}_8 \oplus \mathbb{Z}\overline{(-K_X)})$. The set $\{\overline{E} - \overline{E}_0 \ / \ E$ est une courbe (-1) sur X $\}$ is a sbset of $NS(X)$ and of $K_X^{\perp}$; if M^+ denotes $\{x \in NS(X)$ / pour un certain entier naturel non nul n, $nx \in M\}$, then $M^+ = NS(X) \bigcap K_X^{\perp}$. Since $\dfrac{NS(X)}{M}$ is a finitely generated group, the group $(\dfrac{NS(X)}{M})_t$ whose elements are the torsion elements of $\dfrac{NS(X)}{M}$ is finite ([8, Theorem 8.5., p.46]). Since $(\dfrac{NS(X)}{M})_t = \dfrac{M^+}{M}$, there exists $N \in \mathbb{N}$, $N \neq 0$, $\forall x \in M^+$ $Nx \in M$; in particular there exists some non-negative integer N such that for every (-1)-curve E $N(\overline{E} - \overline{E}_0) \in \mathbb{Z}\overline{C}_1 \oplus \ldots \oplus \mathbb{Z}\overline{C}_8 \oplus \mathbb{Z}\overline{(-K_X)}$.

According to the lemma 5.1 (p. 14) we have for all $i = 1, 2, \ldots, 9$:

$$-E_0.C_i \leq \sum_{j=1}^{j=8} C_i.C_j \alpha_j(E) \leq \frac{1}{a_i} - E_0.C_i.$$

In matrix form:

$$\begin{pmatrix} -E_0.C_1 \\ \vdots \\ -E_0.C_8 \\ 0 \end{pmatrix} \leq \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right) \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_8 \\ \alpha_9 \end{pmatrix} \leq \begin{pmatrix} \frac{1}{a_1} - E_0.C_1 \\ \vdots \\ \frac{1}{a_8} - E_0.C_8 \\ 0 \end{pmatrix}.$$

where $D = (C_i.C_j)_{1 \leq i,j \leq 8}$.

The equality $E^2 = -1$ gives

$$\alpha_1(E)^2 + \cdots + \alpha_8(E)^2 - \sum_{1 \leq i < j \leq 8} C_i.C_j \alpha_i(E)\alpha_j(E) = E_0.C_1\alpha_1(E) + \cdots + E_0.C_8\alpha_8(E) + \alpha_9(E).$$

Since D is a negative definite square matrix of order eight, it is invertible and an application of corollary 2.16 (p. 8) gives the fact that $((\alpha_1(E), \ldots, \alpha_8(E), \alpha_9(E)))_E$ is bounded.

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